Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of MATHEMATICAL

www.elsevier.com/locate/jmaa

On the spectrums of frame multiresolution analyses ☆

Hong Oh Kim [a], Rae Young Kim [a,*], Jae Kun Lim [b]

[a] *Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Guseong-Dong, Yuseong-Gu, Daejeon 305-701, Republic of Korea*
[b] *Department of Computational Mathematics and Informatics, Hankyong National University, 67 Sokjong-Dong, Ansung-City, Kyonggi-Do 456-749, Republic of Korea*

Received 30 April 2004

Available online 11 January 2005

Submitted by L. Grafakos

Abstract

We first give conditions for a univariate square integrable function to be a scaling function of a frame multiresolution analysis (FMRA) by generalizing the corresponding conditions for a scaling function of a multiresolution analysis (MRA). We also characterize the spectrum of the 'central space' of an FMRA, and then give a new condition for an FMRA to admit a single frame wavelet solely in terms of the spectrum of the central space of an FMRA. This improves the results previously obtained by Benedetto and Treiber and by some of the authors. Our methods and results are applied to the problem of the 'containments' of FMRAs in MRAs. We first prove that an FMRA is always contained in an MRA, and then we characterize those MRAs that contain 'genuine' FMRAs in terms of the unique low-pass filters of the MRAs and the spectrums of the central spaces of the FMRAs to be contained. This characterization shows, in particular, that if the low-pass filter of an MRA is almost everywhere zero-free, as is the case of the MRAs of Daubechies, then the MRA contains no FMRAs other than itself.

☆ This work was supported by Korea Research Foundation Grant: KRF-2002-070-C00004.
* Corresponding author.
E-mail addresses: hkim@amath.kaist.ac.kr (H.O. Kim), rykim@amath.kaist.ac.kr (R.Y. Kim), jaekun@hknu.ac.kr (J.K. Lim).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.11.050

Keywords: Multiresolution analysis; Frame; Wavelets; Shift-invariant spaces; Spectrum

1. Introduction

A multiresolution analysis (MRA) was introduced by Mallat [22] and Meyer [23] primarily as a tool to construct and analyze the orthonormal wavelets. Ever since its introduction it has been applied in such diverse fields as subband coding, image compression, mathematical tomography, and the numerical solutions of the partial differential equations [10]. In particular, Daubechies' celebrated constructions of compactly supported orthonormal wavelets with arbitrary regularity used the full structures of MRAs [9]. Then, its generalization, a frame multiresolution analysis (FMRA), was considered and applied in the analysis of narrow band signals with more freedom in the constructions of wavelets with fast iterative structures by Benedetto and Li [1]. This paper is the continuation of our previous works in which various characterizations of the entities comprising an MRA or an FMRA were given [15–17,19,20]. We first characterize the scaling functions and the spectrums of the 'central' space of an FMRA. Then, we give a new condition for an FMRA to admit a single frame wavelet solely in terms of the spectrum of the central space of the FMRA. Other such characterizations in terms of the zero sets of the low-pass filters of an FMRA were given by Benedetto and Treiber [2] and by some of the authors [20], independently, and their generalizations were considered in another article of ours [17]. Our characterizations of the scaling functions of an FMRA and the spectrum of the central space of an FMRA are applied to the problem of the containments of FMRAs in MRAs. In particular, we show that an FMRA is always contained in an MRA. Then the MRAs containing 'genuine' FMRAs are also characterized in terms of the unique low-pass filters of the MRAs and the spectrums of the central spaces of the FMRAs to be contained. The latter characterization shows, in particular, that if the low-pass filter of an MRA is almost everywhere zero-free, as is the case of the MRAs of Daubechies, then the MRA contains no FMRAs other than itself.

Before we go into the details, we introduce some notations which will be used throughout this article. Let $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary dyadic dilation operator such that, for $f \in L^2(\mathbb{R})$,

$$Df(x) := 2^{1/2} f(2x),$$

and let, for each $t \in \mathbb{R}$, $T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary translation operator such that, for $f \in L^2(\mathbb{R})$,

$$T_t f(x) := f(x - t).$$

We now state the definition of the MRA of Mallat and Meyer and that of the FMRA of Benedetto and Li.

Definition 1.1. A family $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ is said to be an *MRA* if

(i) $V_j \subset V_{j+1}$ for each $j \in \mathbb{Z}$;

(ii) $D(V_j) = V_{j+1}$ for each $j \in \mathbb{Z}$;
(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) there exists a *scaling function* $\varphi \in V_0$ such that $\{T_k\varphi\colon k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

On the other hand, $\{V_j\colon j \in \mathbb{Z}\}$ is said to be an FMRA if condition (iv) is replaced by

(v) there exists a *scaling function* $\varphi \in V_0$ such that $\{T_k\varphi\colon k \in \mathbb{Z}\}$ is a tight frame with a frame bound one for V_0.

We refer to [8,10,11,31] for the definitions and the basic properties of frames and Riesz bases of $L^2(\mathbb{R})$. Note that even though an FMRA is more general than an MRA, a modifier is attached to it. The normalizations in conditions (iv) and (v) are not restrictive. It is well known that if the integer shifts of a square integrable function form a Riesz basis (frame) of its closed linear span, then there is another element of the closed linear span such that its integer shifts form an orthonormal basis (tight frame with frame bound one, respectively) for the same closed linear span [4,10,23].

Suppose we are given an MRA with a scaling function φ. Since $\varphi \in V_0 \subset V_1$ and since $\{DT_k\varphi\colon k \in \mathbb{Z}\}$ is an orthonormal basis of V_1, there exists unique $a \in \ell^2(\mathbb{Z})$ such that

$$\varphi = \sum_{k\in\mathbb{Z}} a(k) DT_k\varphi.$$

Taking the Fourier transform of the both sides yields a unique $m \in L^2(\mathbb{T})$ such that

$$\hat{\varphi}(x) = m\left(\frac{x}{2}\right)\hat{\varphi}\left(\frac{x}{2}\right) \quad \text{for a.e. } x \in \mathbb{R}, \tag{1.1}$$

where

$$\mathbb{T} := [-\pi, \pi].$$

This m is called the *low-pass filter* of the MRA with the given scaling function φ.

On the other hand, suppose that we are given an FMRA, and that φ is a scaling function of the FMRA. Then (1.1) still holds, and the low-pass filter m is still an element of $L^2(\mathbb{T})$. The low-pass filter, however, is not unique since the integer shifts of the scaling function are assumed to be a frame, not necessarily a Riesz basis, of its closed linear span [31]. In some situations, the low-pass filter, rather than the scaling function, plays the central role in the theory and the applications of FMRAs [1,2,20]. In this article we are going to elaborate that this non-uniqueness of the low-pass filter does not, in any way, matter in characterizing various aspects of FMRAs since it is the 'spectrum' of the central space of an FMRA, rather than the low-pass filter, that determines the structure of the FMRA.

The article is organized in the following manner: in Section 2, after a brief introduction of notations and conventions, we characterize the scaling functions of an FMRA (Theorem 2.4) and the spectrum of the central space of an FMRA (Theorem 2.6). Then we give another condition for an FMRA to admit a single frame wavelet solely in terms of the spectrum of the central space of an FMRA (Theorem 2.8). Examples illustrating our results are also given. In Section 3, we first show that an FMRA is always contained in an MRA (Theorem 3.2). Then we find the conditions for an MRA to contain an FMRA in terms of the spectrum of the central space of the FMRA to be contained and the unique low-pass filter of the MRA (Theorem 3.3). As a corollary we show that if the unique low-pass filter of an MRA is almost everywhere zero-free, as is the case of the Daubechies' MRAs, then no FMRAs other than itself is contained in the MRA (Corollary 3.4).

2. Scaling functions and spectrums of FMRAs

In this section we characterize the scaling functions of FMRAs (Theorem 2.4) and the spectrums of the central spaces of FMRAs (Theorem 2.6). We then give a new condition for an FMRA to admit a single frame wavelet [2,19,20] in Theorem 2.8. We first fix the notations and introduce some concepts that will be used later.

A **closed** subspace S of $L^2(\mathbb{R})$ is said to be *shift-invariant* if $T_k f \in S$ for any $k \in \mathbb{Z}$ and $f \in S$. We refer to [4,5,12,14,26,30] for the details about the shift-invariant spaces. Let $\Phi \subset L^2(\mathbb{R})$. Then

$$S := \mathcal{S}(\Phi) := \overline{\operatorname{span}}\{T_k\varphi\colon \varphi \in \Phi,\ k \in \mathbb{Z}\}$$

is clearly a shift-invariant subspace of $L^2(\mathbb{R})$. In this case we say that S is the shift-invariant space *generated* by Φ, and Φ is a *generating set* for S. Moreover, it is known that a shift-invariant space always has a countable generating set [4,5,12]. The following form of the Fourier transform is used in this paper: for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $x \in \mathbb{R}$, let

$$\hat{f}(x) := \int_{\mathbb{R}} f(t)e^{-ixt}\,dt.$$

Of course, the Plancherel theorem extends the Fourier transform to a $\sqrt{2\pi}$ times a unitary operator of $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, let

$$\hat{f}_{||x} := \big(\hat{f}(x + 2\pi k)\big)_{k\in\mathbb{Z}},$$

which is in $\ell^2(\mathbb{Z})$ for a.e. $x \in \mathbb{T}$ and, for $A \subset L^2(\mathbb{R})$, let

$$\hat{A}_{||x} := \{\hat{f}_{||x}\colon f \in A\}.$$

For a shift-invariant subspace $S \subset L^2(\mathbb{R})$, the *spectrum* $\sigma(S)$ of S is defined to be

$$\sigma(S) := \big\{x \in \mathbb{T}\colon \hat{S}_{||x} \neq \{0\}\big\}.$$

Moreover, we have the following fundamental result whose proof can be found in [4, Proposition 3.1], [5, Proposition 1.5], [12] or [14, Theorem 2.1].

Proposition 2.1. *Let S be a closed subspace of $L^2(\mathbb{R})$. Then it is shift-invariant if and only if $\hat{S}_{||x}$ is a closed subspace of $\ell^2(\mathbb{Z})$ a.e. $x \in \mathbb{T}$. Let Φ be a countable generating set. Then, for a.e. $x \in \mathbb{T}$,*

$$\hat{S}_{||x} = \overline{\operatorname{span}}\{\hat{\varphi}_{||x}\colon \varphi \in \Phi\}.$$

Moreover, a square integrable function f is in S if and only if $\hat{f}_{||x} \in \hat{S}_{||x}$ a.e. $x \in \mathbb{T}$.

We use the following notational conventions throughout the paper. For $E \subset \mathbb{T}$, we let

$$\tilde{E} := E + 2\pi\mathbb{Z}.$$

If E is a Lebesgue measurable subset of $\mathbb{R}$, then $|E|$ denotes the Lebesgue measure of E. All subsets of $\mathbb{R}$ in this paper, with some exceptions which are clear from the context, are defined modulo Lebesgue null sets, and the containments and equalities among subsets of $\mathbb{R}$ are also in the sense of modulo Lebesgue null sets. We also use the convention that the multiplication of a function f defined on the real line with a function p defined on $\mathbb{T}$ means the multiplication of f with the 2π-periodic extension of p.

The first statement of the following proposition is an almost folklore result. See [4,10]. The proof of the second statement, using different techniques, can be found, for example, in [1,4,6,7,19,26].

Proposition 2.2. *For $f \in L^2(\mathbb{R})$, $\{T_k f\colon k \in \mathbb{Z}\}$ is an orthonormal basis of its closed linear span if and only if*

$$\sum_{k\in\mathbb{Z}} \left|\hat{f}(x + 2\pi k)\right|^2 = 1 \quad \text{for a.e. } x \in \mathbb{T}.$$

It is a tight frame with frame bound one for its closed linear span if and only if

$$\sum_{k\in\mathbb{Z}} \left|\hat{f}(x + 2\pi k)\right|^2 = 1 \quad \text{for a.e. } x \in \mathbb{T} \setminus N,$$

where $N := \{x \in \mathbb{T}\colon\ \hat{f}_{||x} = 0\}$.

For $f \in L^2(\mathbb{R})$, the support of f is defined to be

$$\operatorname{supp}(f) := \left\{x \in \mathbb{R}\colon\ f(x) \neq 0\right\}.$$

Note that it is defined modulo Lebesgue null sets, and hence it is in accordance with our convention. We need the following proposition which is [3, Theorem 4.3].

Proposition 2.3 [3]. *For $\varphi \in L^2(\mathbb{R})$ and $j \in \mathbb{Z}$, let $V_j := \overline{\operatorname{span}}\{D^j T_k \varphi\colon\ k \in \mathbb{Z}\}$. Then $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ if and only if*

$$\bigcup_{j\in\mathbb{Z}} 2^j \operatorname{supp}(\hat{\varphi}) = \bigcup_{j\in\mathbb{Z}} \operatorname{supp}\bigl(\hat{\varphi}\bigl(2^j\ \cdot\bigr)\bigr) = \mathbb{R}. \tag{2.1}$$

The following is a generalization of [13, Theorem 5.2 in Chapter 7]. See also [29]. We present a quick proof of this generalization by using Proposition 2.3.

Theorem 2.4. *For $\varphi \in L^2(\mathbb{R})$ and $j \in \mathbb{Z}$, let $V_j := \overline{\operatorname{span}}\{D^j T_k \varphi\colon\ k \in \mathbb{Z}\}$. Then $\{V_j\colon\ j \in \mathbb{Z}\}$ is an FMRA with a scaling function φ if and only if:*

(1) $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(x + 2\pi k)|^2 = 0$ *or* 1 *for a.e.* $x \in \mathbb{T}$;
(2) *there exists $m \in L^2(\mathbb{T})$, called a low-pass filter, such that $\hat{\varphi}(2x) = m(x)\hat{\varphi}(x)$ for a.e. $x \in \mathbb{R}$;*
(3) $\lim_{j\to\infty} |\hat{\varphi}(2^{-j}x)| = 1$ *for a.e.* $x \in \mathbb{R}$.

Proof. ($\Rightarrow$) (1) follows from Proposition 2.2. In order to prove that (2) holds, we follow the same line of argument (actually, the more or less standard argument) that leads to (1.1). Since $\{T_k\varphi\colon k \in \mathbb{Z}\}$ is a tight frame for V_0 with frame bound one, so is $\{DT_k\varphi\colon k \in \mathbb{Z}\}$ for $V_1 = D(V_0)$. This fact combined with the one that $\varphi \in V_0 \subset V_1$ imply

$$\varphi = \sum_{k\in\mathbb{Z}} \langle \varphi, DT_k\varphi\rangle DT_k\varphi.$$

Moreover, $(\langle \varphi, DT_k\varphi\rangle)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ since $\{DT_k\varphi\colon k \in \mathbb{Z}\}$ is a frame. Hence (2) follows by taking the Fourier transform of the both sides of the above equation. By Proposition 2.3, for a.e. $x \in \mathbb{R}$, there exists $l_x \in \mathbb{Z}$ such that $\hat{\varphi}(2^{l_x}x) \neq 0$. For any $-j < l_x$, we have, by a repeated application of condition (2) of Theorem 2.4,

$$0 < \left|\hat{\varphi}\left(2^{l_x}x\right)\right| = \left(\prod_{k=l_x-1}^{-j} \left|m\left(2^k x\right)\right|\right)\left|\hat{\varphi}\left(2^{-j}x\right)\right|. \tag{2.2}$$

Notice that V_0 is a shift-invariant space be definition. Moreover, by Proposition 2.1,

$$\sigma(V_0) = \left\{x \in \mathbb{T}\colon \left(\hat{\varphi}(x + 2\pi k)\right)_{k\in\mathbb{Z}} \neq 0\right\}. \tag{2.3}$$

Conditions (1) and (2) imply that, for a.e. $x \in \sigma(V_0)$,

$$1 \geqslant \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(2x + 4\pi k)\right|^2 = \left|m(x)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(x + 2\pi k)\right|^2 = \left|m(x)\right|^2. \tag{2.4}$$

Therefore $|m(x)| \leqslant 1$ for a.e. $x \in \sigma(V_0)^\sim$. (2.2) and (2.3) imply that $2^{l_x}x \in \sigma(V_0)^\sim$. (2.2) again implies that $2^k x \in \sigma(V_0)^\sim$ for each $k \leqslant l_x$. Consequently, it implies that $|\hat{\varphi}(2^{-j}x)|$ is eventually non-decreasing, and, hence, converges to a positive number, say, α_x as $j \to \infty$. We have, by condition (1), $\alpha_x \leqslant 1$. We now follow the line of argument in the proof of [13, Theorem 1.7 in Chapter 2]. Since $\{D^j T_k\varphi\colon k \in \mathbb{Z}\}$ is a tight frame with frame bound one, $P_j f := \sum_{k\in\mathbb{Z}} \langle f, D^j T_k\varphi\rangle D^j T_k\varphi$ is an orthogonal projection onto V_j. Let $f := \check{\chi}_{[-1,1]}$, where $\vee$ denotes the inverse Fourier transform. Then $\|P_j f\|^2 \to \|f\|^2 = 1/\pi$ by condition (iii) of Definition 1.1. Let $j \geqslant 1$. Since $P_j f \in V_j$ and since $\{D^j T_k\varphi\colon k \in \mathbb{Z}\}$ is a tight frame with frame bound one for V_j, we have, for j large enough,

$$\begin{aligned}
\|P_j f\|^2 &= \sum_{k\in\mathbb{Z}} \left|\left\langle f, D^j T_k\varphi\right\rangle\right|^2 \\
&= \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}} \left| \int_{-1}^{1} 2^{-j/2}\bar{\hat{\varphi}}\left(2^{-j}x\right) e^{-i2^{-j}kx}\, dx \right|^2 \\
&= 2^j \sum_{k\in\mathbb{Z}} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[-2^{-j},2^{-j}]}(x)\bar{\hat{\varphi}}(x) e^{-ikx}\, dx \right|^2 \\
&= \frac{2^j}{2\pi} \int_{-2^{-j}}^{2^{-j}} \left|\hat{\varphi}(x)\right|^2 dx = \frac{1}{2\pi} \int_{-1}^{1} \left|\hat{\varphi}\left(2^{-j}x\right)\right|^2 dx,
\end{aligned}$$

where the Parseval's theorem is used in the next-to-last equality. Now the dominated convergence theorem implies that $\alpha_x = 1$ for a.e. $x \in \mathbb{R}$.

($\Leftarrow$) (1) and (2) imply conditions (i), (ii) and (v) of Definition 1.1. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ by [3, Corollary 4.14]. Considering Proposition 2.3, we only need to show that $\bigcup_{j\in\mathbb{Z}} 2^j \operatorname{supp}(\hat{\varphi}) = \mathbb{R}$, which follows by (3). $\square$

Suppose that $\{V_j\colon\ j \in \mathbb{Z}\}$ is an FMRA with a scaling function φ. We now characterize the spectrum of V_0 (Theorems 2.6). Recall that (2.3) holds. As in the proof of Theorem 2.4 there exists a low-pass filter $m \in L^2(\mathbb{T})$ satisfying

$$\hat{\varphi}(2x) = m(x)\hat{\varphi}(x) \quad \text{for a.e. } x \in \mathbb{R}. \tag{2.5}$$

We first derive some characterizing properties of $A := \sigma(V_0) \subset \mathbb{T}$. Notice that (2.5) is equivalent to

$$\bigl(\hat{\varphi}(2x + 4\pi k)\bigr)_{k\in\mathbb{Z}} = m(x)\bigl(\hat{\varphi}(x + 2\pi k)\bigr)_{k\in\mathbb{Z}} \quad \text{for a.e. } x \in \mathbb{T}.$$

Recall that $|m(x)| \leqslant 1$ for a.e. $x \in \tilde{A}$ by (2.4).

Now, let $B \subset \mathbb{R}$ be the support of $\hat{\varphi}$, and define

$$B_{\mathbb{T}} := \{x \pmod{2\pi}\colon\ x \in B\} \subset \mathbb{T} = [-\pi, \pi].$$

Then the following should hold:

$$B \subset \tilde{A}; \tag{2.6}$$

$$B_{\mathbb{T}} = A; \tag{2.7}$$

$$\frac{1}{2}B \subset B. \tag{2.8}$$

(2.6) and (2.7) hold by (2.3), and (2.8) by (2.5).

Obviously, (2.7) implies (2.6). Note that the support of m contains $(1/2)B = \operatorname{supp}(\hat{\varphi}(2\cdot))$ by (2.5). Since m is 2π-periodic, $((1/2)B)^{\sim} \subset \operatorname{supp}(m)$. Hence

$$\left(\left(\frac{1}{2}B\right)^{\sim} \cap B\right) \subset \bigl(\operatorname{supp}(m) \cap \operatorname{supp}(\hat{\varphi})\bigr) \subset \operatorname{supp}\bigl(\hat{\varphi}(2\cdot)\bigr) = \frac{1}{2}B,$$

again, by (2.5). Combining this fact with (2.8), we have

$$\frac{1}{2}B = \left(\frac{1}{2}B\right)^{\sim} \cap B. \tag{2.9}$$

Obviously, (2.9) implies (2.8). These facts lead us to:

Theorem 2.5. *Let $A \subset [-\pi, \pi]$. Then there exist $\varphi \in L^2(\mathbb{R}), m \in L^2(\mathbb{T})$ satisfying* (2.5) *with*

$$\sum_{k\in\mathbb{Z}} \bigl|\hat{\varphi}(x + 2\pi k)\bigr|^2 = \chi_{\tilde{A}}(x) \quad \textit{for a.e. } x \in \mathbb{T} \tag{2.10}$$

if and only if there exists $B \subset \mathbb{R}$ satisfying conditions (2.7) *and* (2.9)*. In this case,* (2.6) *and* (2.8) *hold.*

Proof. We only need to show that conditions (2.7) and (2.9) imply the existence of such φ and m. Let $f \in L^2(\mathbb{R})$ be a compactly supported function such that

$$\hat{f}(2x) = n(x)\hat{f}(x)$$

holds for any real x for some trigonometric polynomial n. We may choose, for example, $f := \chi_{[0,1]}$. Then $\hat{f}$ has, being a restriction of an entire function by a well-known theorem of Paley and Wiener [27], at most a countable number of zeros. Note also that n has finite number of zeros. If we define g and p by

$$\hat{g}(x) := \hat{f}(x)\chi_B(x), \qquad p(x) := n(x)\chi_{((1/2)B)^\sim}(x),$$

then, obviously, $g \in L^2(\mathbb{R})$ and p is 2π-periodic. Moreover,

$$\hat{g}(2x) = \hat{f}(2x)\chi_{(1/2)B}(x) = n(x)\chi_{((1/2)B)^\sim}(x)\hat{f}(x)\chi_B(x) = p(x)\hat{g}(x)$$

by (2.9). Since $\hat{f}$ has at most a countable number of zeros, the support of the periodic function

$$\sum_{k\in\mathbb{Z}} \left|\hat{g}(x+2\pi k)\right|^2 = \sum_{k\in\mathbb{Z}} \left|\hat{f}(x+2\pi k)\right|^2 \chi_B(x+2\pi k)$$

is equal to $\tilde{B}$ except possibly for a countable number of points. Note that $\tilde{B} = \tilde{A}$ by condition (2.7).

We define a 2π-periodic function

$$q(x) := \frac{\chi_{\tilde{B}}(x)}{(\sum_{k\in\mathbb{Z}} |\hat{g}(x+2\pi k)|^2)^{1/2}}.$$

By our convention of identifying measurable sets which are different modulo Lebesgue null sets, we have $\operatorname{supp}(q) = \tilde{B}$. Also define φ by

$$\hat{\varphi}(x) := q(x)\hat{g}(x).$$

Notice that $(q(x)/q(x))\chi_{\tilde{B}}(x) = \chi_{\tilde{B}}(x)$. Hence $(q(x)/q(x))\chi_{\tilde{B}}(x)\hat{g}(x) = \hat{g}(x)$ since $\operatorname{supp}(\hat{g}) = B \subset \tilde{B}$. Therefore, we can check that

$$\begin{aligned}\hat{\varphi}(2x) &= q(2x)\hat{g}(2x) = q(2x)p(x)\hat{g}(x) = p(x)q(2x)\frac{q(x)}{q(x)}\chi_{\tilde{B}}(x)\hat{g}(x)\\ &= p(x)\frac{q(2x)}{q(x)}\chi_{\tilde{B}}(x)q(x)\hat{g}(x) = p(x)\frac{q(2x)}{q(x)}\chi_{\tilde{B}}(x)\hat{\varphi}(x).\end{aligned}$$

(2.5) is satisfied since $p(x)q(2x)\chi_{\tilde{B}}(x)/q(x)$ is 2π-periodic. Moreover, (2.10) implies that it is essentially bounded. $\square$

Again, we suppose that $\{V_j\colon j \in \mathbb{Z}\}$ is an FMRA with a scaling function φ. Let $A := \sigma(V_0)$. Since the integer translates of φ are assumed to be a tight frame with frame bound one, we have, by Proposition 2.2, for a.e. $x \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(x+2\pi k)\right|^2 = \chi_{\tilde{A}}(x).$$

Let $x \in \mathbb{R}$. Then $2^{-j}x \in \mathbb{T}$ for sufficiently large $j > 0$. We have

$$\left|\hat{\varphi}\left(2^{-j}x\right)\right|^2 + \sum_{k\neq 0}\left|\hat{\varphi}\left(2^{-j}x + 2\pi k\right)\right|^2 = \chi_A\left(2^{-j}x\right).$$

Since the limit-superior of the left-hand side of the above equation is greater than or equal to 1 as j goes to infinity by Theorem 2.4(3), the right-hand side is greater than $1/2$ for any sufficiently large j. Hence, for a.e. real x,

$$\chi_A\left(2^{-j}x\right) \to 1 \quad \text{as } j \to \infty, \tag{2.11}$$

since $\chi_A(2^{-j}x)$ is 0 or 1 for any j.

Notice that if we let $B := \operatorname{supp}(\hat{\varphi})$, then (2.1) can be rephrased as the following condition:

$$\bigcup_{j\in\mathbb{Z}} 2^j B = \mathbb{R}. \tag{2.12}$$

Combined with Proposition 2.3, Theorem 2.5 implies:

Theorem 2.6. *$A \subset \mathbb{T}$ is the spectrum of the central space V_0 of an FMRA $\{V_j\}_{j\in\mathbb{Z}}$ if and only if there exists $B \subset \mathbb{R}$ satisfying conditions* (2.7), (2.9) *and* (2.12). *In this case,* (2.6), (2.8) *and* (2.11) *hold.*

Any interval of the form $[-a, a]$ $(a \leqslant \pi)$ is easily seen to be the spectrum of the central space of an FMRA. On the other hand, one may check that $[3\pi/4, \pi]$ cannot be the spectrum of the central space of an FMRA. If a subset $B \subset \mathbb{R}$ satisfies (2.6), then $B \subset [3\pi/4, \pi] + 2\pi\mathbb{Z}$. A direct calculation, however, shows that $[3\pi/4, \pi] + 2\pi\mathbb{Z}$ and $[3\pi/8, \pi/2] + \pi\mathbb{Z}$ are disjoint. Hence (2.8) cannot be satisfied.

For a non-trivial set which is the spectrum of an FMRA, we borrow the following example from [17]. We use this example again when we illustrate Theorem 2.8 below. For $2\pi/3 < b < \pi$, let $A := A_{-1} \uplus A_0 \uplus A_1$, where

$$A_{-1} := \left[-\pi, -\frac{2\pi}{3}\right], \qquad A_0 := \left[\frac{b}{2} - \pi, \frac{2\pi}{3}\right], \qquad A_1 := [b, \pi].$$

Here $\uplus$ denotes the disjoint union. We also let $B := A \subset \mathbb{T}$. Then (2.7) is trivially satisfied. Since A_0 contains a neighborhood of the origin, (2.12) is satisfied. Since $b < \pi$, $(1/2)\mathbb{T} \subset A_0 \subset \mathbb{T}$. Hence $(1/2)B \subset B$. Therefore (2.9) is also satisfied.

The choice of B is not unique. The set $C := C_{-1} \uplus C_0 \uplus C_1$ with

$$C_{-1} := \left[b - 2\pi, -\frac{2\pi}{3}\right], \qquad C_0 := \left[\frac{b}{2} - \pi, \frac{2\pi}{3}\right], \qquad C_1 := \left[b, \frac{4\pi}{3}\right],$$

may play the role of B above. This can be verified as follows: since $[b - 2\pi, -\pi] + 2\pi = A_1$ and $[\pi, (4\pi)/3] - 2\pi = A_{-1}$, $C_{\mathbb{T}} = A$. Hence (2.7) is satisfied. (2.9) is also satisfied since $(1/2)C \subset C_0 \subset \mathbb{T}$ and $((1/2)C + 2\pi k) \cap C = \emptyset$ for any non-zero integer k. Finally, C satisfies (2.12) since it contains a neighborhood of 0.

It is shown in [17] that $\check{\chi}_A$ and $\check{\chi}_C$ are the scaling functions of two 'quasi-biorthogonal' FMRAs. Therefore, it is not any wonder that $A = A_{\mathbb{T}} = C_{\mathbb{T}}$ is the spectrum of an FMRA. □

Given an FMRA $\{V_j\colon\ j \in \mathbb{Z}\}$, it may or may not admit a single frame wavelet $\psi \in V_1 \ominus V_0$ such that $\{D^j T_k \psi\colon\ j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ [1,2,19]. The existence and construction of such a single frame wavelet are addressed in [2,19]. It is proved in [19] that there always exist two functions $\psi_1, \psi_2 \in V_1 \ominus V_0$ such that $\{D^j T_k \psi_i\colon\ j,k \in \mathbb{Z},\ i = 1,2\}$ is a frame for $L^2(\mathbb{R})$. The following necessary and sufficient condition for an FMRA to admit a single frame wavelet is obtained in [2,19].

Proposition 2.7 [2,19]. *Suppose that $\{V_j\colon\ j \in \mathbb{Z}\}$ is an FMRA with a scaling function φ. Let m be its low-pass filter. Then there exists a frame wavelet $\psi \in W_0 := V_1 \ominus V_0$ such that $\{D^j T_k \psi\colon\ j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ if and only if $m(x/2)$ and $m(x/2 - \pi)$ are not simultaneously zero a.e. $x \in \Delta_2$, where*

$$\Delta_2 := \left\{x \in \mathbb{T}\colon \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{x}{2} + 2\pi k\right)\right|^2 \neq 0 \text{ and } \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{x}{2} + \pi + 2\pi k\right)\right|^2 \neq 0\right\}.$$

In the above characterization, the condition is given in terms of the non-unique low-pass filter m associated with the given scaling function. Interestingly enough, we are now able to give a new characterization solely in terms of the spectrum of central space of an FMRA.

Theorem 2.8. *Suppose that $\{V_j\colon\ j \in \mathbb{Z}\}$ is an FMRA with $A := \sigma(V_0)$. Then there exists a single frame wavelet in $V_1 \ominus V_0$ such that $\{D^j T_k \psi\colon\ j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ if and only if the set*

$$(\mathbb{T} \setminus A) \cap (2A) \cap \big[(2A - 2\pi) \cup (2A + 2\pi)\big]$$

is a Lebesgue null set.

Proof. Let φ be a scaling function and m be a low-pass filter. Recall that $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(x + 2\pi k)|^2 = \chi_{\tilde{A}}(x)$, for a.e. $x \in \mathbb{R}$. Notice that the set Δ_2 in Proposition 2.7 can be given as

$$\Delta_2 = \mathbb{T} \cap (2A) \cap \big\{(2A - 2\pi) \cup (2A + 2\pi)\big\}.$$

By (2.5), we have

$$\chi_{\tilde{A}}(x) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(x + 2\pi k)|^2 = \left|m\left(\frac{x}{2}\right)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{x}{2} + 2\pi k\right)\right|^2 + \left|m\left(\frac{x}{2} + \pi\right)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{x}{2} + \pi + 2\pi k\right)\right|^2.$$

It is now easy to see that

$$\big\{x \in \Delta_2\colon m(x/2) = 0 = m(x/2 + \pi)\big\} = (\mathbb{T} \setminus A) \cap \Delta_2.$$

The corollary now follows by noting that

$$(\mathbb{T} \setminus A) \cap \Delta_2 = (\mathbb{T} \setminus A) \cap (2A) \cap \big[(2A - 2\pi) \cup (2A + 2\pi)\big]. \qquad \square$$

We illustrate the above theorem by an example. Now let A be as in the example following Theorem 2.6. Direct calculations show that:

$$\begin{aligned}
&\mathbb{T}\setminus A=\left[-\frac{2\pi}{3},\frac{b}{2}-\pi\right]\uplus\left[\frac{2\pi}{3},b\right];\\
&\mathbb{T}\cap(2A)=\mathbb{T};\\
&\mathbb{T}\cap(2A-2\pi)=\left[-\pi,-\frac{2\pi}{3}\right]\uplus[2b-2\pi,0];\\
&\mathbb{T}\cap(2A+2\pi)=\left[0,\frac{2\pi}{3}\right]\uplus[b,\pi].
\end{aligned}$$

Here $\uplus$ denotes the mutually disjoint union. Since $(2\pi)/3<b$, $b/2-\pi<2b-2\pi$. Hence $(\mathbb{T}\setminus A)\cap(2A)\cap[(2A-2\pi)\cup(2A+2\pi)]$ is a Lebesgue null set. Hence the FMRA admits a single frame wavelet by Theorem 2.8.

Similar calculations show that if the central spectrum of an FMRA is $[-a,a]$ with $0<a\leqslant\pi/2$, then it admits a single frame wavelet. On the other hand, if $\pi/2<a\leqslant\pi$, then the FMRA does not admit a single frame wavelet. This recovers the previous results contained in [2,19,20].

3. Containments of FMRAs in MRAs

In this section we show that an FMRA is always contained in an MRA (Theorem 3.2) and characterize the spectrums of the central spaces of FMRAs contained in an MRA (Theorem 3.3). As a corollary we show that if the unique low-pass filter of an MRA with a given scaling function is almost everywhere zero-free, then the MRA contains no FMRAs other than itself. For the precise meaning of containment, we refer to the corresponding theorems. We first state the following straight-forward lemma.

Lemma 3.1. *For $\eta,\varphi\in L^2(\mathbb{R})$, let $\mathcal{V}_0=\overline{\operatorname{span}}\{T_k\eta\colon k\in\mathbb{Z}\}$ and let $V_0:=\overline{\operatorname{span}}\{T_k\varphi\colon k\in\mathbb{Z}\}$. Suppose that $\{T_k\eta\colon k\in\mathbb{Z}\}$ is an orthonormal basis for $\mathcal{V}_0$. Then $V_0\subset\mathcal{V}_0$ and $\{T_k\varphi\colon k\in\mathbb{Z}\}$ is a tight frame with frame bound one for V_0 if and only if*

$$\hat{\varphi}(x)=\lambda(x)\hat{\eta}(x)\quad \textit{for a.e. } x\in\mathbb{R},$$

for some $\lambda\in L^2(\mathbb{T})$ such that $|\lambda(x)|=\chi_{\sigma(V_0)^\sim}(x)$ for $a.e.$ $x\in\mathbb{R}$.

We now show that an FMRA is always contained in an MRA in the following sense. The construction techniques similar to ours in the following proof are found in [13,15,24, 25,29].

Theorem 3.2. *Suppose that $\{V_j\colon j\in\mathbb{Z}\}$ is an FMRA. Then there exists an MRA $\{\mathcal{V}_j\colon j\in\mathbb{Z}\}$ such that $V_j\subset\mathcal{V}_j$ for each $j\in\mathbb{Z}$.*

Proof. Assume that $\{V_j\colon j\in\mathbb{Z}\}$ is an FMRA with a scaling function φ. Note that $\hat{\varphi}_{||x}=0$ for $x\notin\sigma(V_0)$. For $j\geqslant 0$, let

$$E_j:=\{x\in\mathbb{T}\colon \hat{\varphi}_{||2^{-j}x}\neq 0 \text{ and } \hat{\varphi}_{||2^{-m}x}=0,\ 0\leqslant m<j\}.$$

By Theorem 2.4(3), we have $\mathbb{T}=\uplus_{j\geqslant 0}E_j$. Hence, for a.e. $x\in\mathbb{T}$, there exists a unique $j(x)\in\mathbb{N}\cup\{0\}$ such that $x\in E_{j(x)}$. Therefore we have defined a function j such that

$$j:\mathbb{T}\to\mathbb{N}\cup\{0\},$$
$$j(x):=\text{the smallest non-negative integer } l \text{ such that } 2^{-l}x\in\sigma(V_0).$$

For $n\geqslant 0$, define P_n: $\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})$ via

$$(P_n a)(k):=\begin{cases} a(k), & \text{if } k\in n\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases} \tag{3.1}$$

Define

$$\hat\eta_{||x}:=P_{2^{j(x)}}\left(\left(\hat\varphi\left(\frac{x+2\pi k}{2^{j(x)}}\right)\right)_{k\in\mathbb{Z}}\right)$$

for a.e. $x\in\mathbb{T}$. Let $\mathcal{V}_j:=\overline{\operatorname{span}}\{D^jT_k\eta\colon k\in\mathbb{Z}\}$ for $j\in\mathbb{Z}$. Notice that we have defined

$$\hat\eta_{||x}(2^{j(x)}k)=\hat\eta\big(x+2\pi\big(2^{j(x)}k\big)\big)=\hat\varphi\left(\frac{x+2\pi(2^{j(x)}k)}{2^{j(x)}}\right)=\hat\varphi(2^{-j(x)}x+2\pi k),$$

and $\hat\eta_{||x}(k)=0$ if $k\notin 2^{j(x)}\mathbb{Z}$. Hence $\hat\eta_{||x}$ is the 'up-sampled' version of $\hat\varphi_{||2^{-j(x)}x}$, i.e.,

$$\begin{cases} \hat\eta_{||x}(2^{j(x)}k)=\hat\varphi_{||2^{-j(x)}x}(k), & k\in\mathbb{Z},\\ \hat\eta_{||x}(k)=0, & k\notin 2^{j(x)}\mathbb{Z}.\end{cases} \tag{3.2}$$

Therefore, $\|\hat\eta_{||x}\|^2_{\ell}(\mathbb{Z})^2=\|\hat\varphi_{||2^{-j(x)}x}\|^2_{\ell}(\mathbb{Z})^2=1$ for a.e. $x\in\mathbb{T}$. Hence $\eta\in L^2(\mathbb{R})$. By Proposition 2.2, $\{T_k\eta\colon k\in\mathbb{Z}\}$ is an orthonormal basis for $\mathcal{V}_0$. Notice that if $x\in\sigma(V_0)$, then $j(x)=0$. Hence in this case $\hat\varphi_{||x}=\hat\eta_{||x}$. Therefore,

$$\hat\varphi_{||x}=\chi_{\sigma(V_0)^\sim}(x)\hat\eta_{||x}. \tag{3.3}$$

Hence $V_0\subset\mathcal{V}_0$ by Proposition 2.1 since $\operatorname{span}\{\hat\varphi_{||x}\}\subset\operatorname{span}\{\hat\eta_{||x}\}$ a.e. $x\in\mathbb{T}$ by (3.3). Since $V_j=D^j(V_0)$ and $\mathcal{V}_j=D^j(\mathcal{V}_0)$, we have $V_j\subset\mathcal{V}_j$ for $j\in\mathbb{Z}$. To show that $\{\mathcal{V}_j\colon j\in\mathbb{Z}\}$ is an MRA with a scaling function η, we only need to check that η satisfies conditions (2) and (3) of Theorem 2.4 in view of [13, Theorem 5.2 in Chapter 7].

(3.3) implies that $|\hat\varphi(x)|\leqslant|\hat\eta(x)|$ for a.e. $x\in\mathbb{R}$. Condition (3) of Theorem 2.4 implies that

$$1\geqslant\big|\hat\eta\big(2^{-j}x\big)\big|\geqslant\big|\hat\varphi\big(2^{-j}x\big)\big|\to 1,$$

as j tends to infinity for a.e. $x\in\mathbb{R}$. Hence η satisfies condition (3) of Theorem 2.4.

Now we find $m\in L^2(\mathbb{T})$ such that $\hat\eta(2x)=m(x)\hat\eta(x)$ for a.e. $x\in\mathbb{R}$, which is equivalent to

$$\big(\hat\eta(2x+4\pi k)\big)_{k\in\mathbb{Z}}=m(x)\hat\eta_{||x}, \tag{3.4}$$

for a.e. $x\in\mathbb{T}$. Let m^F be a low-pass filter for φ such that

$$\big(\varphi(2x+4\pi k)\big)_{k\in\mathbb{Z}}=m^F(x)\hat\varphi_{||x}$$

for a.e. $x\in\mathbb{T}$. It is rather technical to check condition (2) of Theorem 2.4. We are going to show that if we define 2π-periodic function m via

$$m(x):=\begin{cases} m^F(x), & \text{if } x\in\sigma(V_0)\cap\frac{1}{2}(\sigma(V_0)^\sim),\\ 1, & \text{if } x\in\big(\sigma(V_0)\cap\frac{1}{2}((\mathbb{T}\setminus\sigma(V_0))^\sim)\cap\big[-\frac{\pi}{2},\frac{\pi}{2}\big]\big)\\ & \qquad\uplus\big((\mathbb{T}\setminus\sigma(V_0))\setminus\frac{1}{2}(\sigma(V_0)^\sim)\cap\big[-\frac{\pi}{2},\frac{\pi}{2}\big]\big),\\ 0, & \text{otherwise,}\end{cases}$$

then the condition is satisfied.

Notice that

$$\begin{aligned}\mathbb{T} = {} & \left(\sigma(V_0) \cap \frac{1}{2}\big(\sigma(V_0)\big)^{\sim}\right) \uplus \left(\sigma(V_0) \cap \frac{1}{2}\big(\mathbb{T} \setminus \sigma(V_0)\big)^{\sim}\right) \\ & \uplus \left(\big(\mathbb{T} \setminus \sigma(V_0)\big) \cap \frac{1}{2}\big(\sigma(V_0)\big)^{\sim}\right) \uplus \left(\big(\mathbb{T} \setminus \sigma(V_0)\big) \cap \frac{1}{2}\big(\mathbb{T} \setminus \sigma(V_0)\big)^{\sim}\right).\end{aligned}$$

First, suppose $x \in \sigma(V_0)$ and $2x \in (\sigma(V_0))^{\sim}$. Since $x \in \sigma(V_0)$, $j(x) = 0$. Hence $\hat{\eta}_{||x} = \hat{\varphi}_{||x}$. If $x \in \sigma(V_0) \subset \mathbb{T}$ and $2x \in \sigma(V_0)^{\sim}$, then either $2x \in \sigma(V_0)$ or one of $2x + 2\pi$ and $2x - 2\pi$ is in $\sigma(V_0)$. Suppose that $2x \in \sigma(V_0)$. Then, obviously, $\hat{\eta}(2x + 2\pi k) = \hat{\varphi}(2x + 2\pi k)$ for each integer k since $j(2x) = 0$. Hence,

$$\hat{\eta}(2x + 4\pi k) = \hat{\varphi}(2x + 4\pi k) = m^F(x)\hat{\varphi}(x + 2\pi k) = m^F(x)\hat{\eta}(x + 2\pi k)$$

for each integer k. Suppose, on the other hand, that $2x + 2\pi \in \sigma(V_0)$. Then $j(2x + 2\pi) = 0$. Therefore, for each integer k,

$$\begin{aligned}\hat{\eta}(2x + 4\pi k) &= \hat{\eta}\big(2x + 2\pi + 2\pi(2k - 1)\big) = \hat{\varphi}\big(2x + 2\pi + 2\pi(2k - 1)\big) \\ &= \hat{\varphi}(2x + 4\pi k) = m^F(x)\hat{\varphi}(x + 2\pi k) = m^F(x)\hat{\eta}(x + 2\pi k).\end{aligned}$$

The last equality holds since $j(x) = 0$. The case that $2x - 2\pi \in \sigma(V_0)$ can be handled similarly. We define $m(x) := m^F(x)$ for a.e. $x \in \sigma(V_0) \cap (1/2)(\sigma(V_0))^{\sim}$. Then we have $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = m(x)\hat{\eta}_{||x}$ for a.e. $x \in \sigma(V_0) \cap (1/2)(\sigma(V_0))^{\sim}$.

Secondly, suppose $x \in \sigma(V_0)$ and $2x \in (\mathbb{T} \setminus \sigma(V_0))^{\sim}$, i.e., $j(x) = 0$, and $2x \notin (\sigma(V_0))^{\sim}$. Then $\hat{\eta}_{||x} = \hat{\varphi}_{||x}$. If $|x| \leqslant \pi/2$, then $2x \in \mathbb{T} \setminus \sigma(V_0)$. Thus $j(2x) = 1$. By (3.2), for $k \in \mathbb{Z}$,

$$\hat{\eta}(2x + 4\pi k) = \hat{\eta}_{||2x}(2k) = \hat{\varphi}_{||2^{-1}2x}(k) = \hat{\varphi}(x + 2\pi k) = \hat{\eta}(x + 2\pi k).$$

We define $m(x) := 1$ for a.e. $x \in \sigma(V_0) \cap (1/2)(\mathbb{T} \setminus \sigma(V_0))^{\sim} \cap [-\pi/2, \pi/2]$. Then $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = m(x)\hat{\eta}_{||x}$ for a.e. $x \in \sigma(V_0) \cap (1/2)(\mathbb{T} \setminus \sigma(V_0))^{\sim} \cap [-\pi/2, \pi/2]$.

If $x \in [-\pi, -\pi/2]$, then $2x + 2\pi \in \mathbb{T} \setminus \sigma(V_0)$. Thus $j(2x + 2\pi) \geqslant 1$. Hence $2k - 1 \notin 2^{j(2x+2k\pi)}\mathbb{Z}$ for $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, we have, by (3.2),

$$\hat{\eta}(2x + 4\pi k) = \hat{\eta}(2x + 2\pi + 2\pi(2k - 1)) = \hat{\eta}_{||2x+2\pi}(2k - 1) = 0.$$

Similarly, if $x \in [\pi/2, \pi]$, then $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = 0$. Hence, for $x \in \sigma(V_0) \cap (1/2)(\mathbb{T} \setminus \sigma(V_0))^{\sim} \cap (\mathbb{T} \setminus [-\pi/2, \pi/2])$, we define $m(x) := 0$, which implies $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = 0 = m(x)\hat{\eta}_{||x}$.

Thirdly, suppose $x \in \mathbb{T} \setminus \sigma(V_0)$ and $2x \in (\sigma(V_0))^{\sim}$. Then, either $2x \in \sigma(V_0)$ or one of $2x + 2\pi$ and $2x - 2\pi$ is in $\sigma(V_0)$. Suppose that $2x \in \sigma(V_0)$. Then, $\hat{\eta}(2x + 4\pi k) = \hat{\varphi}(2x + 4\pi k)$ for each integer k. Hence,

$$\hat{\eta}(2x + 4\pi k) = \hat{\varphi}(2x + 4\pi k) = m^F(x)\hat{\varphi}(x + 2\pi k) = 0$$

for each integer k. Suppose, on the other hand, that $2x + 2\pi \in \sigma(V_0)$. We now have

$$\begin{aligned}\hat{\eta}(2x + 4\pi k) &= \hat{\eta}\big(2x + 2\pi + 2\pi(2k - 1)\big) = \hat{\varphi}\big(2x + 2\pi + 2\pi(2k - 1)\big) \\ &= \hat{\varphi}(2x + 4\pi k) = m^F(x)\hat{\varphi}(x + 2\pi k) = 0\end{aligned}$$

for each integer k since $x \in \mathbb{T} \setminus \sigma(V_0)$. Similarly, if $2x - 2\pi \in \sigma(V_0)$, then $\hat{\eta}(2x + 4\pi k) = 0$ for each integer k. So we define $m(x) := 0$ for a.e. $x \in (\mathbb{T} \setminus \sigma(V_0)) \cap (1/2)(\sigma(V_0))^{\sim}$. Then we have $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = m(x)\hat{\eta}_{||x}$ for a.e. $x \in (\mathbb{T} \setminus \sigma(V_0)) \cap (1/2)(\sigma(V_0))^{\sim}$.

Finally, let $x \in \mathbb{T} \setminus \sigma(V_0)$ and $2x \notin (\sigma(V_0))^{\sim}$. Notice that if $x \in [-\pi/2, \pi/2]$, then we have $j(2x) = j(x) + 1$. Hence, for each integer k,

$$\begin{aligned}\hat{\eta}\big(2x + 2 \cdot 2\pi 2^{j(x)} k\big) &= \hat{\eta}\big(2x + 2\pi 2^{j(2x)} k\big) = \hat{\eta}_{||2x}\big(2^{j(2x)} k\big) \\ &= \hat{\varphi}_{||2^{-j(2x)}2x}(k) = \hat{\varphi}_{||2^{-j(x)}x}(k) \\ &= \hat{\eta}_{||x}\big(2^{j(x)} k\big) = \hat{\eta}\big(x + 2\pi 2^{j(x)} k\big).\end{aligned}$$

Note that, for $k \notin 2^{j(x)}\mathbb{Z} = 2^{j(2x)}(1/2)\mathbb{Z}$, $2k \notin 2^{j(2x)}\mathbb{Z}$. Hence

$$\hat{\eta}(2x + 2\pi 2k) = 0 = \eta(x + 2\pi k).$$

If we define $m(x) := 1$ in this case, then we have, for a.e. $x \in (\mathbb{T} \setminus \sigma(V_0)) \cap (1/2) \times (\mathbb{T} \setminus \sigma(V_0))^{\sim} \cap [-\pi/2, \pi/2]$, $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = m(x)\hat{\eta}_{||x}$.

If $x \in [-\pi, -\pi/2]$, then $j(2x + 2\pi) \geqslant 1$. Thus $2k - 1 \notin 2^{j(2x+2\pi)}\mathbb{Z}$ for each integer k. Hence we have, for $k \in \mathbb{Z}$,

$$\hat{\eta}(2x + 4\pi k) = \hat{\eta}\big(2x + 2\pi + 2\pi(2k - 1)\big) = \hat{\eta}_{||2x+2\pi}(2k - 1) = 0$$

by (3.2). Similarly, if $x \in [\pi/2, \pi]$, then $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = 0$. Hence, for a.e. $x \in (\mathbb{T} \setminus \sigma(V_0)) \cap (1/2)(\mathbb{T} \setminus \sigma(V_0))^{\sim} \cap (\mathbb{T} \setminus [-\pi/2, \pi/2])$, we take $m(x) := 0$ in this case, which implies $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = 0 = m(x)\hat{\eta}_{||x}$.

We have shown that if we define m as we did, then $(\hat{\eta}(2x + 4\pi k))_{k\in\mathbb{Z}} = m(x)\hat{\eta}_{||x}$ for a.e. $x \in \mathbb{T}$. Hence condition (2) of Theorem 2.4 is satisfied. $\square$

We have seen that an FMRA is always contained in an MRA. It is natural to ask: Does an MRA always contain a 'genuine' FMRA? The corollary to the following theorem (Corollary 3.4) shows that it does not. The following theorem characterizes the spectrums of the central spaces of FMRAs contained in an MRA.

Theorem 3.3. *Let $A \subset \mathbb{T}$. Suppose that $\{\mathcal{V}_j\colon j \in \mathbb{Z}\}$ is an MRA with a scaling function η. Let m be its unique low-pass filter and $N_m := \{x \in \mathbb{R}\colon m(x) = 0\}$. Then there exists an FMRA $\{V_j\colon j \in \mathbb{Z}\}$ with $A = \sigma(V_0)$ such that $V_j \subset \mathcal{V}_j$ for each $j \in \mathbb{Z}$ if and only if*

$$(\mathbb{R} \setminus \tilde{A}) \subset \left(\mathbb{R} \setminus \frac{1}{2}\tilde{A}\right) \cup N_m; \tag{3.5}$$

$$\lim_{j\to\infty} \chi_A\big(2^{-j}x\big) = 1 \quad \textit{for a.e. } x \in \mathbb{R}. \tag{3.6}$$

Proof. ($\Rightarrow$) Since $\{V_j\colon j \in \mathbb{Z}\}$ is an FMRA, there exist a low-pass filter $m^F \in L^2(\mathbb{T})$ and a scaling function φ such that

$$\hat{\varphi}(2x) = m^F(x)\hat{\varphi}(x). \tag{3.7}$$

Since $V_0 \subset \mathcal{V}_0$, Lemma 3.1 implies that

$$\hat{\varphi}(x) = \lambda(x)\hat{\eta}(x) \quad \text{for a.e. } x \in \mathbb{T},$$

for some $\lambda \in L^2(\mathbb{T})$ such that $|\lambda(x)| = \chi_{\tilde{A}}(x)$ for a.e. $x \in \mathbb{T}$. Combining this with (3.7), we have, for a.e. $x \in \mathbb{R}$,

$$\hat{\varphi}(2x) = \lambda(2x)\hat{\eta}(2x) = \lambda(2x)m(x)\hat{\eta}(x) \quad \text{and}$$
$$\hat{\varphi}(2x) = m^F(x)\hat{\varphi}(x) = m^F(x)\lambda(x)\hat{\eta}(x).$$

Notice that $\lambda(2x)$ is π-periodic. Therefore, we have

$$\begin{aligned}\chi_{\frac{1}{2}\tilde{A}}(x)|m(x)|^2 &= \left|\lambda(2x)m(x)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\eta}(x+2\pi k)\right|^2 \\ &= \left|\lambda(x)m^F(x)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\eta}(x+2\pi k)\right|^2 \\ &= \chi_{\tilde{A}}(x)\left|m^F(x)\right|^2,\end{aligned}$$

where the second equality holds by calculating $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(2(x+2\pi k))|^2$ in two ways using the above equations, and the first and the third hold since $|\lambda(x)| = \chi_{\tilde{A}}$ and since $\sum_{k\in\mathbb{Z}} |\hat{\eta}(x+2\pi k)|^2 = 1$ a.e. x. This implies condition (3.5). Condition (3.6) follows by Theorem 2.6.

($\Leftarrow$) Define φ via $\hat{\varphi}(x) := \chi_{\tilde{A}}(x)\hat{\eta}(x)$ and let $V_j := \overline{\text{span}}\{D^j T_k\varphi\colon\ k \in \mathbb{Z}\}$ for $j \in \mathbb{Z}$. It follows from Lemma 3.1 that $V_0 \subset \mathcal{V}_0$. Hence $V_j \subset \mathcal{V}_j$ for $j \subset \mathbb{Z}$. To show that $\{V_j\colon\ j \in \mathbb{Z}\}$ is an FMRA, we only need to check conditions (1) $\sim$ (3) of Theorem 2.4. Proposition 2.2 implies that

$$\sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(x+2\pi k)\right|^2 = \chi_{\tilde{A}}(x).$$

This shows that condition (1) of Theorem 2.4 holds; and also shows that $\sigma(V_0) = A$. Since $\{\mathcal{V}_j\colon\ j \in \mathbb{Z}\}$ is an MRA, we have $\lim_{j\to\infty} |\hat{\eta}(2^{-j}x)| = 1$ for a.e. $x \in \mathbb{R}$ by Theorem 2.4. Combing this with (3.6) yields condition (3) of Theorem 2.4. Notice that, for a.e. $x \in \mathbb{R}$,

$$\hat{\varphi}(2x) = \chi_{\tilde{A}}(2x)\hat{\eta}(2x) = \chi_{\tilde{A}}(2x)m(x)\hat{\eta}(x). \tag{3.8}$$

Define

$$m^F(x) := \begin{cases} \chi_{\tilde{A}}(2x)m(x), & x \in \tilde{A}, \\ 0, & \text{otherwise}. \end{cases}$$

If $x \in \tilde{A}$, then $\hat{\varphi}(x) = \hat{\eta}(x)$; and hence $\hat{\varphi}(2x) = m^F(x)\hat{\varphi}(x)$. If $x \notin \tilde{A}$, then

$$\hat{\varphi}(2x) = \chi_{\tilde{A}}(2x)m(x)\hat{\eta}(x) = \chi_{(1/2)\tilde{A}}(x)m(x)\hat{\eta}(x) = 0$$

by (3.5). Recall that $m^F(x) = 0$ for $x \notin \tilde{A}$. Hence we have

$$\hat{\varphi}(2x) = 0 = m^F(x)\hat{\varphi}(x).$$

This shows that condition (2) of Theorem 2.4 holds. □

It is interesting to note that conditions (3.5) and (3.6) imply the existence of such a set B as in Theorem 2.6. Actually, we could have proved the 'if' part of the above theorem by resorting to Theorem 2.6 in the following way: suppose we are given an MRA $\{\mathcal{V}_j\colon\ j \in \mathbb{Z}\}$

with a scaling function η. Let m, N_m and A be as in Theorem 3.3. Suppose that they satisfy (3.5) and (3.6). A scrutiny of the proof of the 'if' part of the theorem shows that

$$B := \tilde{A} \cap \operatorname{supp}(\hat{\eta})$$

is a candidate. We now show that B satisfy (2.7), (2.9) and (2.12). Since $(\operatorname{supp}(\hat{\eta}))_{\mathbb{T}} = \mathbb{T}$, (2.7) is satisfied. Now suppose that $x \in (1/2)B$. Then $2x \in B = \tilde{A} \cap \operatorname{supp}(\hat{\eta})$. Since $0 \neq \hat{\eta}(2x) = m(x)\hat{\eta}(x)$, $x \notin N_m$ and $x \in \operatorname{supp}(\hat{\eta})$. Suppose that $x \notin \tilde{A}$. Then by (3.5), $2x \notin \tilde{A}$ since $x \notin N_m$. Since $2x$ is assumed to be in $B = \tilde{A} \cap \operatorname{supp}(\hat{\eta})$, the contradiction shows that $x \in \tilde{A}$. Therefore $x \in B$. We have shown that $(1/2)B \subset B$, thereby showing that $(1/2)B \subset ((1/2)B)^{\sim} \cap B$. Suppose, on the other hand, that $x \in ((1/2)B)^{\sim} \cap B$. Then there exists $k_x \in \mathbb{Z}$ such that

$$2x + 4\pi k_x \in B = \tilde{A} \cap \operatorname{supp}(\hat{\eta}), \tag{3.9}$$

$$x \in B = \tilde{A} \cap \operatorname{supp}(\hat{\eta}). \tag{3.10}$$

(3.9) implies that $0 \neq \hat{\eta}(2x + 4\pi k_x) = m(x)\hat{\eta}(x + 2\pi k_x)$. This shows that $m(x) \neq 0$. Since $\hat{\eta}(x) \neq 0$ by (3.10), $\hat{\eta}(2x) = m(x)\hat{\eta}(x) \neq 0$. (3.9) also implies that $2x \in \tilde{A}$. Therefore $2x \in B$. This establishes (2.9). (3.6) and condition (3) of Theorem 2.4 (applied to η) imply that (2.12) is satisfied. This completes the proof of the 'if' part of Theorem 3.3 by Theorem 2.6. $\square$

The ergodicity argument used in the following corollary may also be seen in [15,18,21].

Corollary 3.4. *Suppose that $\{\mathcal{V}_j\colon\ j \in \mathbb{Z}\}$ is an MRA and η its scaling function. Let m be its unique low-pass filter and let $N_m := \{x \in \mathbb{R}\colon\ m(x) = 0\}$ be its zero set. Suppose also that*

$$|N_m| = 0. \tag{3.11}$$

Then the MRA contains no FMRAs other than itself.

Proof. Suppose that an FMRA $\{V_j\colon\ j \in \mathbb{Z}\}$ with a scaling function φ is contained in the MRA. We first show that the FMRA is actually an MRA. Let $A := \sigma(V_0)$. Note that $(N_m)_{\mathbb{T}} = \{x \in \mathbb{T}\colon\ m(x) = 0\}$. A is clearly not an empty set. It suffices to show that $A = \mathbb{T}$ by Proposition 2.2. If we suppose otherwise, then $|\mathbb{T} \setminus A| > 0$. Obviously, $|(N_m)_{\mathbb{T}} \setminus A| = 0$, since $|N_m| = 0$. Then $(\mathbb{T} \setminus A) \subset (\mathbb{T} \setminus (N_m)_{\mathbb{T}})$. Recall our convention that all inclusions are modulo measure zero sets. Now condition (3.5) implies that

$$\mathbb{T} \setminus A \subset \frac{1}{2}(\mathbb{T} \setminus A)^{\sim}. \tag{3.12}$$

Let $T : \mathbb{T} \to \mathbb{T}$ be the Baker's map defined via $Tx := 2x \pmod{2\pi}$. This map is well-known to be measure-preserving, i.e., $|T^{-1}(B)| = |B|$ for any measurable subset B of $\mathbb{T}$, and ergodic [28, Theorem 1.15]. Let $C := \mathbb{T} \setminus A \subset \mathbb{T}$. (3.12) implies that $T(C) \subset C$. Hence we have

$$C \subset T^{-1}\big(T(C)\big) \subset T^{-1}(C). \tag{3.13}$$

Since T is measure-preserving, the Lebesgue measure of $T^{-1}(C)$ equals that of C. Hence $T^{-1}(C) = C$. Since T is ergodic, $C = \emptyset$ or $\mathbb{T}$ [28, Theorem 1.5]. However, $C \neq \emptyset$ since it is assumed to have positive measure. Hence, $C = \mathbb{T}$ and, therefore, $A = \emptyset$, which is a contradiction.

Now $\varphi \in V_0 \subset \mathcal{V}_0$. Since $\{T_k\eta\colon k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{V}_0$, there exists a 2π-periodic function a such that

$$\hat{\varphi}(x) = a(x)\hat{\eta}(x) \quad \text{for a.e. } x \in \mathbb{R}.$$

Since $\{T_k\varphi\colon k \in \mathbb{Z}\}$ is also an orthonormal basis for V_0, we have, by Proposition 2.2,

$$1 = \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(x + 2\pi k)\right|^2 = \left|a(x)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\eta}(x + 2\pi k)\right|^2 = \left|a(x)\right|^2$$

for a.e. $x \in \mathbb{R}$. This shows that a and $1/a$ are in $L^\infty(\mathbb{T})$. Hence $\hat{\eta}(x) = (1/a(x))\hat{\varphi}(x)$ for a.e. $x \in \mathbb{R}$. This implies that $\eta \in V_0$ by Proposition 2.1. Therefore $V_0 = \mathcal{V}_0$, whence $V_j = \mathcal{V}_j$ for each integer j by dilation. □

Recall that the low-pass filter of any compactly supported refinable function satisfies condition (3.11). In particular, the MRAs of Daubechies in [9,10] contain no FMRAs other than themselves. The authors do not know any direct proof of this fact.

Acknowledgments

The authors thank the anonymous referees whose careful and valuable comments helped us to improve this paper.

References

[1] J.J. Benedetto, S. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. 5 (1998) 398–427.
[2] J.J. Benedetto, O.M. Treiber, Wavelet frames: multiresolution analysis and extension principle, in: L. Debnath (Ed.), Wavelet Transforms and Time-Frequency Signal Analysis, Birkhäuser, Boston, 2000, pp. 3–36.
[3] C. de Boor, R. DeVore, A. Ron, On the construction of multivariate (pre)wavelets, Constr. Approx. 9 (1993) 123–166.
[4] C. de Boor, R. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994) 37–78.
[5] M. Bownik, The structure of shift-invariant subspaces of $L^2(\mathbb{R}^d)$, J. Funct. Anal. 177 (2000) 282–309.
[6] P.G. Casazza, O. Christensen, N. Kalton, Frames of translates, Collect. Math. 52 (2001) 35–54.
[7] D.-R. Chen, On the splitting trick and wavelet frame packets, SIAM J. Math. Anal. 31 (2000) 726–739.
[8] C.K. Chui, An Introduction to Wavelets, Academic Press, San Diego, 1992.
[9] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1988) 909–996.
[10] I. Daubechies, Ten Lectures on Wavelets, CBMS–NSF Regional Conf. Ser. in Appl. Math., SIAM, Philadelphia, PA, 1992.
[11] C. Heil, D. Walnut, Continuous and discrete wavelet transforms, SIAM Rev. 31 (1989) 628–666.
[12] H. Helson, Lectures on Invariant Subspaces, Academic Press, New York, 1964.
[13] E. Hernández, G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton, FL, 1996.
[14] R.-Q. Jia, Shift-invariant spaces and linear operator equations, Israel J. Math. 103 (1998) 258–288.
[15] H.O. Kim, R.Y. Kim, Y.H. Lee, J.K. Lim, On Riesz wavelets associated with multiresolution analyses, Appl. Comput. Harmon. Anal. 13 (2002) 138–150.

[16] H.O. Kim, R.Y. Kim, J.K. Lim, Characterizations of biorthogonal wavelets which are associated with biorthogonal multiresolution analyses, Appl. Comput. Harmon. Anal. 11 (2001) 263–272.
[17] H.O. Kim, R.Y. Kim, J.K. Lim, Quasi-biorthogonal frame multiresolution analyses and wavelets, Adv. Comput. Math. 18 (2003) 269–296.
[18] H.O. Kim, R.Y. Kim, J.K. Lim, Local analysis of frame multiresolution analysis with a general dilation matrix, Bull. Austral. Math. Soc. 67 (2003) 285–295.
[19] H.O. Kim, J.K. Lim, Frame multiresolution analysis, Comm. Korean Math. Soc. 15 (2000) 285–308.
[20] H.O. Kim, J.K. Lim, On frame wavelets associated with frame multi-resolution analysis, Appl. Comput. Harmon. Anal. 10 (2001) 61–70.
[21] H.O. Kim, J.K. Lim, Applications of shift-invariant space theory to some problems of multi-resolution analysis of $L^2(\mathbb{R}^d)$, in: D. Deng, D. Huang, R.-Q. Jia, W. Lin, J. Wang (Eds.), Wavelet Analysis and Applications, in: Stud. Adv. Math., vol. 25, Amer. Math. Soc./International Press, Boston, 2001, pp. 183–191.
[22] S. Mallat, Multiresolution approximations and wavelet orthonormal basis of $L^2(\mathbb{R})$, Trans. Amer. Math. Soc. 315 (1989) 69–87.
[23] Y. Meyer, Wavelets and Operators, Cambridge Univ. Press, Cambridge, UK, 1992.
[24] M. Paluszyňsky, H. Šikič, G. Weiss, S. Xiao, Tight frame wavelets, their dimension functions, MRA tight frame wavelets and connectivity properties, Adv. Comput. Math. 18 (2003) 297–327.
[25] M. Papadakis, On the dimension functions of orthonormal wavelets, Proc. Amer. Math. Soc. 128 (2000) 2043–2049.
[26] A. Ron, Z. Shen, Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$, Canad. J. Math. 47 (1995) 1051–1094.
[27] W. Rudin, Real and Complex Analysis, third ed., McGraw–Hill, New York, 1986.
[28] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag, New York, 1982.
[29] X. Wang, The study of wavelets from the properties of their Fourier transforms, PhD thesis, Washington University in St. Louis, 1995.
[30] G. Weiss, E.N. Wilson, The mathematical theory of wavelets, in: Twentieth Century Harmonic Analysis—a celebration, Il Ciocco, 2000, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 33, Kluwer Academic, Dordrecht, 2001, pp. 329–366.
[31] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.